Semester 2 Examination 2016

Question/answer booklet

HALE
SCHOOL

Circle your teacher's initials

JIB	MAW	MPC	SWA	VMU

Mathematics Methods Units 3 & 4

Section Two
(Calculator Assumed)

Your name: M. Key

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for paper: one hundred minutes

$\overline{}$
95

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction tape/fluid, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	**12**	**12**	**100**	**95**	**65**
				Total	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2016.* Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.
     Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

5. The Formula Sheet is **not** handed in with your Question/Answer Booklet.

Question 9 **(9 marks)**

The velocity of a particle is given by $v(t) = 1+0.02t^3 - 0.1t^2$ m/sec.

a) Determine the instantaneous rate of change of velocity at $t=8$. (2 marks)

$a(t) = v'(t) = 0.06t^2 - 0.2t$

$\therefore \quad a(8) = 2.24 \text{ ms}^{-2}$

✓ subs $t=8$ in $\frac{dv}{dt}$ ✓ ans

b) Using the small changes technique, obtain an estimate for the change in velocity from $t=8$ to $t=8.1$ seconds. (3 marks)

Now $\delta v \approx \frac{dv}{dt} \times \delta t$

$= (0.06t^2 - 0.2t) \times \delta t$ ✓ correct formula used

$= (0.06(8)^2 - 0.2(8)) \times 0.1$ ✓ subs $t=8$, $\delta t = 0.1$

$= 0.224 \text{ ms}^{-1}$ ✓ calcs small change

c) Determine the average rate of change of velocity for the period of time from $t=8$ to $t=8.1$. (2 marks)

$\frac{v(8.1)-v(8)}{0.1} = \frac{5.06782 - 4.84}{0.1} = 2.278 \text{ ms}^{-2}$

✓ writes ave change formula

✓ evaluates ave rate of change correctly

d) Calculate, correct to the nearest 0.01 metres, the change in displacement, from $t=5$ to $t=10$ seconds. (2 marks)

$\int_5^{10} x'(t)\,dt = \int_5^{10} 1 + 0.02t^3 - 0.1t^2\,dt$ ✓ integrates $\frac{dx}{dt}$

$= 22.71 \text{ m}$ (2dp) ✓ determines change in displacement correctly

Question 10 **(12 marks)**

James, a second hand car sales manager, realises that 6% of the vehicles for sale in his yards are defective in some minor way. He is prepared to fix all defects but only if the customer returns with a problem. Assume all customers with a problem return for assistance and that X, representing the number of vehicles returned for repairs per month, is a binomial random variable.

a) Determine the probability that if 21 cars are sold during the month:

i) none will be returned.

(2 marks)

$X \sim B(21, 0.06)$ ✓recognises binomial distribution

$P(X=0) = 0.2727$ (4dp) ✓correct probability

ii) no more than three will be returned.

(2 marks)

$X \sim B(21, 0.06)$ ✓recognises binomial distribution

$P(X \leq 3) = 0.9659$ (4dp) ✓correct probability

iii) no more than three will be returned if one has already been returned.

(2 marks)

$$P(X \leq 3 \mid X \geq 1) = \frac{P(1 \leq X \leq 3)}{P(X \geq 1)} = \frac{0.693...}{0.727...}$$

✓uses $P(A|B)$ formula correctly

$= 0.9531$ (4dp) ✓correct probability

Accept alternative interpretation
(where it is known which car is returned)
$\therefore Y \sim B(20, 0.06)$
$P(Y \leq 2) = 0.8850$ (4 d.p.)

Question 10 (cont.)

b) What is the maximum number of cars that they can sell so the probability that at most 5 vehicles are returned is greater than 0.99?

(3 marks)

$Y \sim B(n, 0.06)$ ✓ defines new binomial variable

and $P(Y \le 5) > 0.99$ ✓ writes correct probability statement

(CAS) max $n = 31$

✓ correct answer

y1=binomialCDf(0, 5, x, 0.06)

x	y1
30	0.9921
31	0.9906
32	0.9891
33	0.9873
34	0.9853
35	0.9832

N.B. Table of values working is OK

c) Jenny, a sales manager at James' main competitor, has a similar problem. You are given that the expected number of vehicles returned for repairs per month is 2.03 and that this can be modelled by a binomial random variable with a variance of 1.8879. Calculate how many cars she sold during the month and what proportion of them have defects.

(3 marks)

$E(X) = np = 2.03$

$V(X) = np(1-p) = 1.8879$ ✓ sets up simultaneous equations using given info

Solving (CAS) gives $p = 0.07$ and $n = 29$ ✓ solves for p, n

i.e. she sold 29 cars during the month, 7% of which had defects ✓ interprets answer in context

Question 11 **(8 marks)**

A random survey was conducted to estimate the proportion of WA voters who preferred the Liberal Party or Labor Party for the upcoming state election. It was found that 340 out of 638 people surveyed preferred the Labor Party.

a) Determine a point estimate for p of those who preferred the Labor Party.

(1 mark)

$$\hat{p} = \frac{340}{638} \approx 0.5329\ldots$$

✓ correct answer

b) Use $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ to calculate an 80% confidence interval for p.

(3 marks)

Using $\hat{p} \pm z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

i.e. $\dfrac{340}{638} \pm 1.28\ldots\sqrt{\dfrac{\frac{340}{638}\left(1-\frac{340}{638}\right)}{638}}$

✓ $z = 1.28\ldots$

✓ subs $\hat{p}$ and n into formula correctly

$\therefore 0.5329\ldots \pm 1.28\ldots \times 0.01975\ldots$

or $0.5329\ldots \pm 0.0253\ldots$

i.e. $0.5076 < p < 0.5582$ ✓ correct C.I.

c) A second sample consisting of 300 people provided a confidence interval of $0.49 \leq p \leq 0.61$. Determine the point estimate for p in this sample and the level of confidence for this interval.

(4 marks)

$$\hat{p} = \frac{0.49 + 0.61}{2} = 0.55$$

and $E = \dfrac{0.61 - 0.49}{2} = 0.06$

✓ $\hat{p}$ and E determined

i.e. $0.06 = z\sqrt{\dfrac{0.55(1-0.55)}{300}}$

(CAS) $z = 2.0889\ldots$

✓ determines correct z value

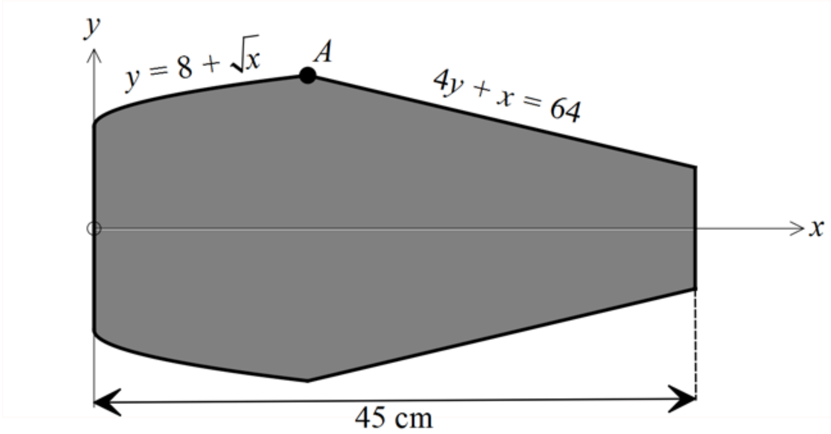
(CAS) $P(-2.0889\ldots < Z < 2.0889\ldots) = 0.963\ldots$ ✓ determines probability

$\therefore$ approx 96.3% confidence

✓ correct confidence level calculated

Question 12 **(5 marks)**

A new type of rowing oar is to be tested. The shape of the blade is as shown shaded in the diagram below.



The y-axis forms the left hand boundary, the x-axis is a line of symmetry and 1 cm = 1 unit on each axis. Point A is where the equations $y = 8 + \sqrt{x}$ and $4y + x = 64$ meet.

a) Determine the coordinates of point A.

(1 mark)

$A(16, 12)$ ✓ correct coordinate

b) Hence, or otherwise, determine the shaded area correct to the nearest square centimetre.

(4 marks)

✓ recognises and uses symmetry ✓ integrates correct functions ✓ correct limits

$$\text{Shaded area} = 2 \times \left(\int_0^{16} 8 + \sqrt{x}\, dx + \int_{16}^{45} \frac{64 - x}{4}\, dx \right)$$

$$= 827.08\dot{3}$$

i.e. shaded area approx $827\ cm^2$ ✓ correct area to nearest cm^2

Question 13 **(5 marks)**

Iodine-131 is present in radioactive waste from the nuclear power industry.

It has a half-life of eight days. This means that every eight days, one half of the iodine-131 decays to a form that is not radioactive.

This decay can be represented by the equation $N = N_0 e^{kt}$,

where N = amount of iodine-131 present after t days, and

N_0 = amount of iodine-131 present initially.

a) Determine the value of *k* correct to three decimal places. (3 marks)

$0.5 N_0 = N_0 e^{8k}$ ✓ eqn for k

i.e. $0.5 = e^{8k}$

(CAS) $k = -0.08664\ldots$ ✓ solves for k

$\therefore$ $k = -0.087$ (3dp) ✓ correct to 3dp

b) If 125 milligrams of iodine-131 are considered to be safe, how many complete days will it take for 78 grams of iodine-131 to decay to a safe amount.

(2 marks)

$0.125 = 78 e^{-0.087t}$ ✓ states equation with correct unit conversion

(CAS) $t = 74.283\ldots$ ✓ solves for t

$\therefore$ it will be safe after approx 75 days

Accept 74 or 75 days

```
solve(0.125=78e^(-.087x)
                {x=73.97873987}
solve(0.5=e^(8x)
                {x=-0.08664339757}
solve(0.125=78e^(-0.0866433
                {x=74.28321775}
```

Question 14 **(9 marks)**

The owners of a chain of discount camping stores plan to open a new shop in Osborne Park. To gauge interest in such a store, they randomly selected 650 people in the area and surveyed them. 390 people were in favour of the new store.

a) Calculate the sample proportion of people who were in favour of the new store. (1 mark)

$\hat{p} = \frac{390}{650} = 0.6$ ✓ans

b) (i) Use the sample proportion calculated in part a) to determine the 95% confidence interval for the population proportion. (2 marks)

(CAS) $0.5623 < p < 0.6377$

✓lower limit ✓upper limit

```
[390, 650, 0.95]⇒[N, n, c]
                 [390 650 0.95]
−invNormCDf(0, c, 1, 0)⇒z
                    1.959963985
N/n⇒p
                            0.6
√(p×(1−p)/n)⇒se
                  0.01921537846
z×se⇒E
                  0.03766144972
[p−E, p+E]
[0.5623385503 0.63766144▶
fRound(ans, 4)
                [0.5623 0.6377]
```

(ii) Interpret the meaning of the confidence interval determined in part b)(i). (2 marks)

✓95% of such intervals

We expect approximately 95% of the interval estimates to contain the actual population proportion p.

✓contain p

(iii) State the margin of error of the confidence interval determined in part b)(i). (1 mark)

$E = 0.03766$ (4dp) ✓ans

c) Use the sample proportion calculated in part a) to determine the sample size required to establish the proportion of within 2% for the 95% confidence interval. (3 marks)

i.e. $0.02 > 1.96\sqrt{\frac{0.6(1-0.6)}{n}}$ ✓sets up inequality correctly

(CAS) $n > 2304.875\ldots$ ✓solves for n

i.e. require sample size of at least 2305

✓rounds up to determine correct sample size

```
invNormCDf("C", 0.95, 1, 0)
                  −1.959963985
solve(0.02>1.959963985×√(0▶
              {n>2304.875293}
```

Question 15 **(7 marks)**

In Australia, size 10 shoes should be between 27.4 cm and 27.8 cm in length. A shoe manufacturer has calculated that its machinery, when set to size 10, produces shoes that are normally distributed with a mean of 27.62 cm and a standard deviation of 0.115 cm.

a) What percentage of shoes produced will be within the size 10 range? (2 marks)

$X \sim N(27.62, 0.115^2)$

$P(27.4 < X < 27.8) = 0.9134$ (4dp)

✓ calcs prob for correct bounds ✓ Correct probability

b) To test the operation of the machine, ten shoes are randomly selected each hour. If two or more are found to be outside the acceptable range, the machine is serviced. What is the probability that a service will be necessary? (3 marks)

$Y \sim B(10, 0.9134)$ ✓ states binomial with correct parameters

$P(Y \leq 8) = 0.2127$ (4d.p.)

✓ recognises up to 8 within range ✓ correct probability

c) The manufacturer would prefer that 95% of the shoes produced be within the acceptable range. To achieve this, the machine will be adjusted to have a normal distribution with a mean of 27.6 cm and a new standard deviation. What standard deviation will be required? (2 marks)

i.e. $1.96\sigma = 0.2$ ✓ $E = 0.2$

$\sigma = 0.1020$ (4dp)

✓ correct σ

Question 16 **(7 marks)**

The thickness x (in microns) of a protective coating applied to a conductor designed to work in corrosive conditions is known to follow a uniform distribution in the interval $20 \le x \le 40$.

a) State the probability density function $f(x)$ for the random variable X.

(2 marks)

$$f(x) = \begin{cases} \frac{1}{20} & 20 \le x \le 40 \\ 0 & \text{otherwise} \end{cases}$$

✓ function ✓ domain

b) Determine the probability that the thickness of the protective coating is exactly 25 microns.

(1 mark)

$P(X = 25) = 0$ ✓ answer

c) Determine the probability that the thickness of the protective coating is less than 36 microns thick given that it is at least 28 microns thick.

(2 marks)

$$P(X < 36 | X > 28) = \frac{P(28 < X < 36)}{P(X > 28)}$$

✓ uses $P(A|B)$ correctly

$$= \frac{8}{12}$$

$$= 2/3$$

✓ correct answer

d) Determine the cumulative distribution function $F(x)$ of the thickness of the protective coating.

(2 marks)

$$F(x) = \int_{20}^{x} \frac{1}{20}\, dt$$

✓ integrates correctly

$$= \left[\frac{t}{20} \right]_{20}^{x}$$

$$= \frac{x}{20} - 1$$

✓ $F(x)$ correct

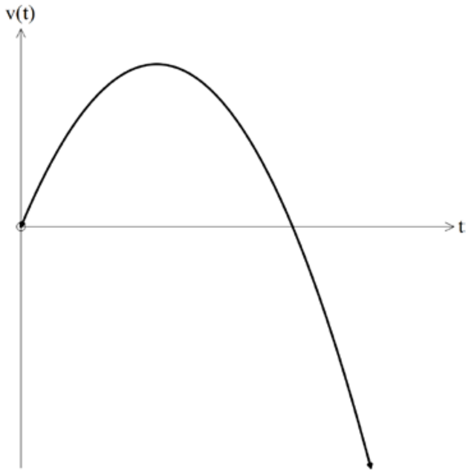
$\int_{20}^{x} 1/20 \, dt$

$\frac{x}{20} - 1$

Question 17 **(8 marks)**

The acceleration, $a(t)$ ms^{-2}, of an object moving in a straight line is given by $a(t) = At + B$, where A and B are non-zero constants.

The velocity-time graph of the object is given below.



a) (i) Given the object is initially at rest, determine the velocity of the particle in terms of A and B.

(2 marks)

$v(t) = \int a(t)\,dt = \int At + B\,dt = \frac{1}{2}At^2 + Bt + C$ ✓ eqn for $v(t)$

$= \frac{1}{2}At^2 + Bt$ since $v(0) = 0$

(ii) Given the object is again at rest after 10 seconds, use your velocity function from part a)(i) to determine B in terms of A.

(1 mark)

since $v(10) = 0$ $50A + 10B = 0$ $\therefore B = -5A$

✓ uses $v(10) = 0$ correctly

Question 17 (cont.)

b) If the object returns to its initial position after T seconds, determine T.

(2 marks)

$x(t) = \int v(t)\,dt = \frac{1}{6}At^3 + \frac{1}{2}Bt^2 + D$

since $x(T) = x(0) = D$, $\frac{1}{6}AT^3 + \frac{1}{2}BT^2 = 0$ ✓ correct eqn involving T

$\therefore$ using ①, $\frac{1}{6}AT^3 - \frac{5}{2}AT^2 = 0$

i.e. $\frac{1}{6}AT^2(T-15) = 0$

since $T \neq 0, A \neq 0$ $\therefore T = 15$ ✓ solves for T

c) Evaluate A and B, given that the acceleration is positive initially and that the object travels a distance of 1000 metres in the first 15 seconds. (3 marks)

The object starts at one point, moves forward for 10s then returns to starting point after 15s.

i.e. travels 500 m in 1st 10 secs

$\therefore x(10) - x(0) = \frac{1}{6}A(10)^3 + \frac{1}{2}B(10)^2 = 500$ ✓ eqn in A, B

i.e. $\frac{1000}{6}A - 250A = 500$

i.e. $A = -6$, $B = 30$

✓ solves for A ✓ solves for B

Question 18 **(9 marks)**

The probability density function of X is given by:

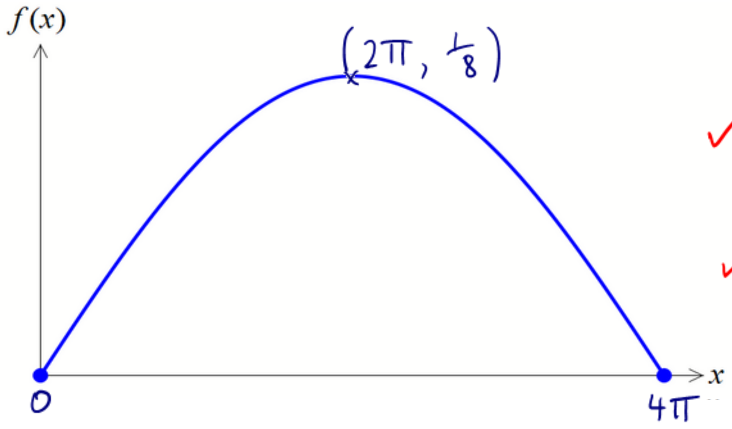
$$f(x) = \begin{cases} \dfrac{1}{a}\sin\left(\dfrac{x}{4}\right) & \text{for } 0 \le x \le 4\pi \\ 0 & \text{otherwise} \end{cases}$$

a) Determine the value of a. (1 mark)

$a = 8$ ✓ans

b) Sketch the graph of $f(x)$ for $0 \le x \le 4\pi$, labelling all key features.

(2 marks)



✓ smooth + accurate curve

✓ key features labelled

Question 18 (cont.)

c) Determine the median and the upper quartile of X.

(3 marks)

$$\int_0^m f(x)\,dx = 0.5 \Rightarrow m = 2\pi = 6.28\ldots$$ ✓ Q_2

$$\int_0^{Q_3} f(x)\,dx = 0.75 \Rightarrow Q_3 = \frac{8\pi}{3} = 8.37\ldots$$

✓ cumulative area = 0.75 ✓ Q_3

Define $f(x)=\frac{1}{8}\sin\left(\frac{x}{4}\right)$

done

$\text{solve}\left(\int_0^m f(x)dx=0.5, m, 0, 0, 4\cdot\pi\right)$

{m=6.283185307}

$\text{solve}\left(\int_0^{uq} f(x)dx=0.75, uq, 0, 0, 4\cdot\pi\right)$

{uq=8.37758041}

d) Determine the mean and variance of X.

(3 marks)

$$E(X) = \int_0^{4\pi} x f(x)\,dx$$
$$= 2\pi \quad (= 6.28\ldots)$$ ✓ correct $E(X)$

$$Var(X) = \int_0^{4\pi} x^2 f(x)\,dx - E(x)^2$$
$$= 8\pi^2 - 32 - (2\pi)^2$$ ✓ uses definition correctly
$$= 4\pi^2 - 32$$ ✓ correct $Var(X)$
$$(= 7.47\ldots)$$

$\int_0^{4\pi} x\times f(x)dx$

$2\cdot\pi$

$\int_0^{4\pi} x^2\times f(x)dx$

$8\cdot\pi^2-32$

$\int_0^{4\pi} x^2\times f(x)dx-(2\cdot\pi)^2$

$4\cdot\pi^2-32$

$\int_0^{4\pi} x^2\times f(x)dx-(2\cdot\pi)^2$

7.478417604

Question 19 **(8 marks)**

The mean μ and standard deviation σ of the uniform distribution on the interval $[a,b]$ are given by:

$$\mu=\frac{a+b}{2} \text{ and } \sigma=\frac{b-a}{2\sqrt{3}}.$$

A calculator can generate random numbers that are uniformly distributed between 0 and 1.

a) For this distribution of the random numbers generated by the calculator, calculate:

(i) the mean (1 mark)

Mean = 0·5 ✓ans

(ii) the standard deviation (correct to three decimal places). (1 mark)

s.d. = 0·289 (3dp) ✓ans

b) What is the probability that a randomly-generated number lies between $\frac{1}{4}$ and $\frac{1}{3}$? (1 mark)

$\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$ or $0{\cdot}08\dot{3}$ ✓ans

c) What is the probability that a randomly-generated number contains no eight in its first six decimal places? (1 mark)

$0{\cdot}9^6$ ✓ ($=0{\cdot}531441$)
ans

Question 19 (cont.)

d) What is the probability that a randomly-generated number contains at most three odd digits in its first six decimal places? Give your answer to four decimal places.

(2 marks)

$X \sim B(6, 0.5)$ ✓ states binomial distribution with correct parameters

$P(X \leq 3) = 0.6563$ (4dp)

✓ correct probability

e) Another uniform distribution on an interval $[a, b]$ has a standard deviation of $2\sqrt{3}$. How wide is the interval?

(2 marks)

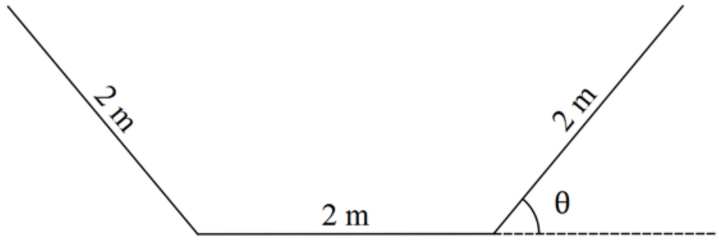
$\dfrac{b-a}{2\sqrt{3}} = 2\sqrt{3}$ $\quad \therefore \quad b - a = (2\sqrt{3})^2 = 12$

✓ sets up equation

✓ correct answer

Question 20 **(8 marks)**

The cross-section of a storm drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown.



a) Show that the area of the trapezium is given by $A = 4\sin\theta(1+\cos\theta)$.

(3 marks)

$\sin\theta = \frac{y}{2} \quad \therefore y = 2\sin\theta$ ✓ determines ⊥ height using θ

$\cos\theta = \frac{x}{2} \quad \therefore x = 2\cos\theta$

$A = \frac{1}{2}(a+b)h$

$= \frac{1}{2}(2 + (2 + 2x))y$ ✓ subs x, y (or similar) into trapezium formula

$= \frac{1}{2}(2 + (2 + 2.2\cos\theta))2\sin\theta$ ✓ formula for A involving θ

$= \sin\theta(4 + 4\cos\theta)$

$= 4\sin\theta(1 + \cos\theta)$

Question 20 (cont.)

b) Use a calculus method to determine the angle θ which maximises the cross-sectional area and hence find this maximum area. (5 marks)

(CAS) $\frac{dA}{d\theta} = 4(\cos\theta + \cos 2\theta)$ ✓ $\frac{dA}{d\theta}$

$\frac{d}{d\theta}(A(\theta))$

$4\cdot(\cos(\theta))^2 - 4\cdot(\sin(\theta))^2 + 4\cdot\cos(\theta)$

simplify(ans)

$4\cdot(\cos(\theta)+\cos(2\cdot\theta))$

max when $\frac{dA}{d\theta} = 0$ ✓ max when $\frac{dA}{d\theta} = 0$

(CAS) $\theta = \frac{\pi}{3}$ (or $1.047...^c$) ✓ solves for θ $(0 < \theta < \frac{\pi}{2})$

Test max

θ	1	1.047...	1.1
$\frac{dA}{d\theta}$	0.5	0	−0.54
	/	—	\

✓ Tests nature (using sign test or second derivative)

(or $A''(\frac{\pi}{3}) = -10.39$)

i.e. max when $\theta = \frac{\pi}{3}$

when area $= 3\sqrt{3}\ m^2$ or $5.20\ m^2$ (2dp)

✓ states correct answer

Define $A(\theta)=4\sin(\theta)\times(1+\cos(\theta))$

done

Define $f(\theta)=\frac{d}{d\theta}(A(\theta))$

done

solve$(4\cdot(\cos(\theta)+\cos(2\cdot\theta))=0,\theta,\pi/4,0,\pi/2)$

$\{\theta=1.047197551\}$

f(1

0.4966218773

f(1.1

−0.5396199833

$A(\pi/3$

5.196152423

END OF SECTION TWO

Additional Working Space

Question Number:___________